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Bifurcation diagrams involving the linear integral of Yehia

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Abstract

In the general case the Hamiltonian system with three degrees of freedom describing the motion of a rigid body in two constant forces does not admit any symmetry groups. Yehia (1986 *Mech. Res. Commun.* **13** 169–72) found conditions under which the equations of motion of the Kowalevski-type top have an integral linear in angular velocities in addition to the energy integral. Later it was noticed that such an integral exists for the same force field for any dynamically symmetric top with the center of force applications in the equatorial plane. Thus, the corresponding system is the natural mechanical system with S^1-symmetry and Smale's program of topological analysis can be fulfilled. Here we construct the bifurcation diagrams of the momentum map for this system and present various types of diagrams depending on one physical parameter.

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1. Introduction

The notion of a mechanical system with symmetry was introduced by Smale [1]. Let a Lie group G (the symmetry group) act on a smooth Riemannian manifold M (the configuration space) and, with tangent maps, on TM (the phase space) preserving the given potential $U : M \to \mathbb{R}$ and the kinetic energy $K : TM \to \mathbb{R}$ $(2K(v) = |v|^2)$. It is supposed that this action generates the principal G-bundle $M \to N = M/G$. Let us call such a symmetry a regular one. For regularity it is sufficient that the action of G on M is proper and free.

The symmetry gives rise to the integral map $L \times H : TM \to \mathbb{R}^k$ $(k = \dim G + 1)$ called the momentum map. Here $H = K + U \circ p_M$ is the total energy (the Hamiltonian function of the system) and L is the momentum integral [1] generalizing the classical notion of a cyclic integral.

For the systems with symmetry Smale formulated the problem of investigation of the phase topology and proved a series of general theorems about the structure of bifurcation diagrams and the topology of integral manifolds,

$$J_{h,\ell} = \{\zeta \in TM : H(\zeta) = h, L(\zeta) = \ell\}, \qquad \hat{J}_{h,\ell} = \{\zeta \in TN : H_\ell(\zeta) = h\}, \tag{1.1}$$

of the initial and reduced systems. These statements essentially use the symmetry's regularity (the regularity of the integral L as a function on TM) and the possibility of defining globally the so-called reduced potential $U_\ell : N \to \mathbb{R}$.

In the dynamics of a rigid body (or, in a more general case, of a gyrostat) the regular symmetry takes place for the problems with axially symmetric potentials. The classical examples are the potentials of the gravity force and of the central Newtonian force. Here $M = SO(3)$, $G = S^1$, $N = S^2$. The latter manifold is called the Poisson sphere and represents all possible positions of the force symmetry axis unit vector with respect to the body. Bifurcation diagrams of the arising momentum map and the corresponding phase topology were investigated by many authors. It is necessary to note here the article of Katok [2], which was published along with the Russian translation of Smale's work, the series of basic publications of Tatarinov [3, 4]. Gashenenko [5–7] gave the most complete description of the variety of existing cases. The problem of full classification of all diagrams and integral manifolds for the classical axially symmetric forces appeared to be verycomplex due to the large number of independent essential parameters.

What happens if the symmetry group action is proper but not free? The orbit space then does not become a manifold but a stratified space [8]. This kind of symmetries are considered in connection with Hamiltonian systems having non-holonomic constraints and nonlinear control systems [9, 10]. The momentum integral in this case is not regular everywhere and the reduced potential on the orbit space is not defined globally. Nevertheless, the results of Smale remain applicable for the regular values of the integral L.

The corresponding examples in the rigid body dynamics were not yet considered, though the problem with a singular symmetry is known. It is the case found by Yehia [11] for the top of Kowalevski type rotating in a double homogeneous force field of the special sort. Now it is known that the Yehia integral is of more general origin and it exists in the wide class of problems of motion of a dynamically symmetric gyrostat [12].

Let $G = S^1$ be the subgroup in $M = SO(3)$. To be definite, suppose that it is the group of rotations around the third coordinate axis $G = \{T(\tau)\}$:

$$T(\tau) = \begin{Vmatrix} \cos\tau & \sin\tau & 0 \\ -\sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

Consider the action of G on M by the inner automorphisms

$$g_\tau(Q) = T(\tau)QT(-\tau), \qquad Q \in SO(3). \tag{1.2}$$

This action is proper (because $SO(3)$ is compact) but not free since all the points of the subgroup G itself are fixed points. Take the orbit space of the action (1.2), identify all fixed points with one point F and introduce the quotient topology. Thereby the obtained set N is homeomorphic to S^2, although the fibre bundle $M \to N$ is not locally trivial in the neighborhood of $F \in N$. The elements of this sphere do not have any clear physical interpretation as it was with the Poisson sphere. But one can easily introduce the spherical coordinates on N expressed, for example, in terms of the appropriately chosen Euler angles.

Usually, the rigid body configuration is represented by the direction cosine matrix Q. The rows of Q are the components of the vectors of some orthonormal inertial basis with respect to the so-called moving frame

$$Oe_1e_2e_3. \tag{1.3}$$

Here O is the body's point fixed in space, $\boldsymbol{e}_j$ $(j = 1, 2, 3)$ are unit mutually orthogonal vectors fixed in the body. Then the action (1.2) expanded to $TSO(3)$ with tangent maps preserves the kinetic energy provided that the axis Oe_3 is the symmetry axis of the inertia tensor $\boldsymbol{I}$ at the point O (the axis of dynamic symmetry).

Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be independent vectors fixed in space. We represent these vectors by their components in the basis (1.3). Then the potential energy can be expressed as a function of $\boldsymbol{\alpha}, \boldsymbol{\beta}$. In the linear approximation one can write

$$U = -(\boldsymbol{r}_1 \cdot \boldsymbol{\alpha} + \boldsymbol{r}_2 \cdot \boldsymbol{\beta}), \tag{1.4}$$

where $\boldsymbol{r}_1, \boldsymbol{r}_2$ are vectors fixed in the body, the dot stands for the standard scalar product in $\mathbb{R}^3$. It is the potential of a superposition of two force fields with the constant intensities $\boldsymbol{\alpha}, \boldsymbol{\beta}$. This field is called a double homogeneous field. The function (1.4) will be preserved by the transformations (1.2) if we suppose that

$$\boldsymbol{r}_1 = \boldsymbol{e}_1, \qquad \boldsymbol{r}_2 = \boldsymbol{e}_2, \tag{1.5}$$

$$|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|, \qquad \boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0. \tag{1.6}$$

The corresponding momentum integral (the Hamilton function of the group action) has the form

$$L = \boldsymbol{M} \cdot (\boldsymbol{\gamma} - |\boldsymbol{\alpha}|^2 \boldsymbol{e}_3), \tag{1.7}$$

where $\boldsymbol{M}$ is the kinetic momentum vector and $\boldsymbol{\gamma} = \boldsymbol{\alpha} \times \boldsymbol{\beta}$. The integral (1.7) was first found by Yehia [11] for the generalized Kowalevski case in which the principal moments of inertia satisfy the ratio 2:2:1. But we see that this integral exists for any dynamically symmetric top satisfying (1.5) and (1.6). Let n denote the ratio of the equatorial inertia moment to the axial one. The value n is therefore the unique essential dimensionless parameter of this problem.

All singular points of the action of G on TM lie on the zero level of the integral L. Therefore, for any non-zero constant ℓ, we obtain the principal G-bundle

$$L^{-1}(\ell) = \{\zeta \in TM : L(\zeta) = \ell\} \to T(N\backslash\{F\}) \cong T\mathbb{R}^2.$$

The reduced system is then defined in the usual way; the integral manifolds (1.1) are constructed by Smale's method. The classification of integral manifolds is based on the study of bifurcation diagrams of corresponding momentum maps.

In the present work we construct a one-parameter family of bifurcation diagrams for a class of problems with potential functions (1.4) admitting the symmetry (1.2) with singularities. We point out the corresponding critical motions of the top. Also the values of the parameter n are found which separate different types of diagrams. These values may be the source of some partial integrable cases. Possible ways of further investigation and generalizations are discussed.

2. Parametrical reduction of equations of motion and first integrals

The problem of motion of a rigid body in force fields with potentials of the type (1.4) was first formulated in the fundamental work of Bogoyavlensky [13]. The physical model is a heavy magnet moving in the gravitational and the constant magnetic fields. The equations of motion referred to the moving frame are

$$\begin{aligned} &\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} = \boldsymbol{M}\times\boldsymbol{\omega} + \boldsymbol{r}_1\times\boldsymbol{\alpha} + \boldsymbol{r}_2\times\boldsymbol{\beta}, \\ &\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t} = \boldsymbol{\alpha}\times\boldsymbol{\omega}, \qquad \frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}t} = \boldsymbol{\beta}\times\boldsymbol{\omega}. \end{aligned} \tag{2.1}$$

Here $\boldsymbol{\omega} = \boldsymbol{M}\boldsymbol{I}^{-1}$ is the angular velocity. The vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ (the fields intensities) are constant in space as shown by the second group of equations (2.1). The vectors $\boldsymbol{r}_1, \boldsymbol{r}_2$ constant in the body can be treated as the radius vectors of the center of the field applications. In the future it is convenient to consider all vectors as rows. Then tensors are placed at the right of vectors.

The restriction of system (2.1) to any non-degenerate common level $P^6(a, b, c)$ of three geometrical integrals

$$|\boldsymbol{\alpha}|^2 = a^2, \qquad |\boldsymbol{\beta}|^2 = b^2, \qquad \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = c \qquad (|c| < ab) \tag{2.2}$$

in $\mathbb{R}^9(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{M})$ is Hamiltonian with three degrees of freedom with respect to the Lie–Poisson brackets [13] having the geometrical integrals as Casimir functions. The Hamilton function is

$$H = \tfrac{1}{2}\boldsymbol{M}\cdot\boldsymbol{\omega} - \boldsymbol{r}_1\cdot\boldsymbol{\alpha} - \boldsymbol{r}_2\cdot\boldsymbol{\beta}. \tag{2.3}$$

Choose a basis of the principal inertia axes for the moving frame (1.3). Suppose that the inertia tensor has the symmetry axis $O\boldsymbol{e}_3$, the ratio n of the equatorial inertia moment to the axial one is arbitrary, the radius vectors of the centers of the fields application are also arbitrary with the only condition that they are parallel to the equatorial plane

$$\boldsymbol{r}_1\cdot\boldsymbol{e}_3 = 0, \qquad \boldsymbol{r}_2\cdot\boldsymbol{e}_3 = 0. \tag{2.4}$$

Let D be a non-degenerate 2×2-matrix. The transformation

$$\begin{Vmatrix}\boldsymbol{r}_1\\ \boldsymbol{r}_2\end{Vmatrix} \mapsto D\begin{Vmatrix}\boldsymbol{r}_1\\ \boldsymbol{r}_2\end{Vmatrix}, \qquad \begin{Vmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{Vmatrix} \mapsto (D^{-1})^T\begin{Vmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{Vmatrix}, \qquad \boldsymbol{M}\mapsto\boldsymbol{M}$$

preserves (2.1) and (2.3), thus leading to an equivalent system. Constant symmetric matrices

$$R = \begin{Vmatrix}\boldsymbol{r}_1\cdot\boldsymbol{r}_1 & \boldsymbol{r}_1\cdot\boldsymbol{r}_2\\ \boldsymbol{r}_2\cdot\boldsymbol{r}_1 & \boldsymbol{r}_2\cdot\boldsymbol{r}_2\end{Vmatrix}, \qquad A = \begin{Vmatrix}\boldsymbol{\alpha}\cdot\boldsymbol{\alpha} & \boldsymbol{\alpha}\cdot\boldsymbol{\beta}\\ \boldsymbol{\beta}\cdot\boldsymbol{\alpha} & \boldsymbol{\beta}\cdot\boldsymbol{\beta}\end{Vmatrix}^{-1}$$

change as follows: $R\mapsto DRD^T$, $A\mapsto DAD^T$. There exists $D\in GL(2,\mathbb{R})$ such that R becomes the identity matrix and A becomes diagonal ($c = 0$). Obviously, the property (2.4) also holds for the new orthonormal pair $\boldsymbol{r}_1, \boldsymbol{r}_2$. Then $\boldsymbol{r}_1, \boldsymbol{r}_2$ can be chosen as the principal inertia basis in the equatorial plane giving the relations (1.5). Thus, we bring the pairs $\boldsymbol{r}_1, \boldsymbol{r}_2$ and $\boldsymbol{\alpha}, \boldsymbol{\beta}$, respectively, to orthonormal and orthogonal ones. This process was first presented in [14] and is known as the parametrical reduction for two constant fields.

Suppose that *after* the reduction in addition to the fact of mutual orthogonality of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we obtain $a = b$. Then we come to the conditions (1.6) without the constraints of Kowalevski type for the moments of inertia. The action (1.2) with respect to the variables $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{M}$ becomes

$$g_\tau\left(\begin{Vmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\\ \boldsymbol{M}\end{Vmatrix}\right) = T(\tau)\begin{Vmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\\ \boldsymbol{M}\end{Vmatrix}T(-\tau). \tag{2.5}$$

It preserves system (2.1), the Hamilton function H and also the time-invariant manifold $P^6(a, a, 0)$. The corresponding momentum integral coincides with (1.7) [12].

Choose in (2.1), (2.2) the measurement units to obtain $\boldsymbol{I} = \mathrm{diag}\{n, n, 1\}$, $a = 1$. From now on, we write P^6 for $P^6(1, 1, 0)$. The first integrals of (2.1) on P^6 are

$$H = \tfrac{1}{2}\left[n\left(\omega_1^2 + \omega_2^2\right) + \omega_3^2\right] - \alpha_1 - \beta_2,$$
$$L = n[\omega_1(\alpha_2\beta_3 - \alpha_3\beta_2) + \omega_2(\alpha_3\beta_1 - \alpha_1\beta_3)] + \omega_3(\alpha_1\beta_2 - \alpha_2\beta_1 - 1).$$

For given $\boldsymbol{\alpha}, \boldsymbol{\beta}$ let Q be the matrix with rows $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\alpha} \times \boldsymbol{\beta}$. Obviously, the map $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{M}) \mapsto (Q, \boldsymbol{\omega})$ is the diffeomorphism of P^6 onto $TSO(3)$. System (2.1) restricted to P^6 is therefore a natural mechanical system on $SO(3)$ with S^1-symmetry having the set of singular points $\boldsymbol{\alpha} \times \boldsymbol{\beta} = \boldsymbol{e}_3$. Denote by Σ the bifurcation diagram of the momentum map

$$J = L \times H : P^6 \to \mathbb{R}^2.$$

Recall that by definition Σ consists of the points $(\ell, h) \in \mathbb{R}^2$ over which J fails to be locally trivial. Thus, when (ℓ, h) crosses Σ, the integral manifolds (1.1) undertake topological changes. Therefore, finding Σ is a necessary part of the topological analysis of the problem. Due to the compact character of the function H, the diagram Σ coincides with the set of critical values of J. First, we discuss the set of critical points of the integrals L and H. In the case of regular symmetry this set is trivial; it contains only the possible equilibria of the top. As it is shown in the next section, the points of singularity add some other solutions.

3. Critical points of the first integrals

We use the change of variables first introduced in [15] and generalizing the change of Kowalevski to the case of two constant fields

$$\begin{aligned} &x_1 = (\alpha_1 - \beta_2) + \mathrm{i}(\alpha_2 + \beta_1), && x_2 = (\alpha_1 - \beta_2) - \mathrm{i}(\alpha_2 + \beta_1),\\ &y_1 = (\alpha_1 + \beta_2) + \mathrm{i}(\alpha_2 - \beta_1), && y_2 = (\alpha_1 + \beta_2) - \mathrm{i}(\alpha_2 - \beta_1),\\ &z_1 = \alpha_3 + \mathrm{i}\beta_3, && z_2 = \alpha_3 - \mathrm{i}\beta_3,\\ &w_1 = \omega_1 + \mathrm{i}\omega_2, && w_2 = \omega_1 - \mathrm{i}\omega_2,\\ &w_3 = \omega_3. \end{aligned} \tag{3.1}$$

Here $\mathrm{i}^2 = -1$. Conditions (2.2) on P^6 take the form

$$z_1^2 + x_1y_2 = 0, \qquad z_2^2 + x_2y_1 = 0, \qquad x_1x_2 + y_1y_2 + 2z_1z_2 = 4. \tag{3.2}$$

Introduce the variables x, y, z such that

$$x^2 = x_1x_2, \qquad y^2 = y_1y_2, \qquad z^2 = z_1z_2,$$

and apply the following agreement about the signs

$$x \geqslant 0, \qquad \mathrm{sgn}\, y = \mathrm{sgn}\,\mathrm{Re}(y_i). \tag{3.3}$$

Then

$$z^2 = \pm xy, \qquad (x \pm y)^2 = 4, \qquad x \in [0, 2]. \tag{3.4}$$

While investigating critical points of various functions on P^6, in order to avoid the undefined Lagrange multipliers for constraints (3.2), we use the equations proposed in the work [16].

Lemma. *Let f be a smooth function of the complex variables* (3.1). *The critical points of the restriction of f to the submanifold defined by* (3.2) *are described by the system of equations*

$$\begin{aligned} &\partial_{w_1} f = 0, \qquad \partial_{w_2} f = 0, \qquad \partial_{w_3} f = 0,\\ &\left(2z_2\partial_{x_2} + 2z_1\partial_{y_2} - x_1\partial_{z_1} - y_1\partial_{z_2}\right) f = 0,\\ &\left(2z_1\partial_{x_1} + 2z_2\partial_{y_1} - x_2\partial_{z_2} - y_2\partial_{z_1}\right) f = 0,\\ &\left(x_1\partial_{x_1} - x_2\partial_{x_2} + y_1\partial_{y_1} - y_2\partial_{y_2}\right) f = 0. \end{aligned} \tag{3.5}$$

Denote by $\mathcal{C}$ the set of critical points of J. Then $\mathcal{C} = \mathcal{C}^0 \cup \mathcal{C}^1$, where

$$\mathcal{C}^i = \{\zeta \in P^6 : \operatorname{rank} J(\zeta) = i\}.$$

First consider the critical points of the Hamilton function H, which are the equilibria of system (2.1). Equations (3.5) with $f = H$ give

$$w_1 = w_2 = w_3 = 0, \qquad z_1 = z_2 = 0, \qquad y_1 = y_2.$$

It follows from (3.4) that in this case

$$xy = 0. \tag{3.6}$$

If $x = 0$, then $y_1 = y_2 = \pm 2$. We obtain two equilibria

$$\begin{aligned} &\boldsymbol{\omega} = 0, \qquad \boldsymbol{\alpha} = \boldsymbol{e}_1, \qquad \boldsymbol{\beta} = \boldsymbol{e}_2; \\ &\boldsymbol{\omega} = 0, \qquad \boldsymbol{\alpha} = -\boldsymbol{e}_1, \qquad \boldsymbol{\beta} = -\boldsymbol{e}_2. \end{aligned}$$

It is easily checked that both of them are non-degenerate; the first equilibrium is stable, while the second one is unstable. Corresponding values of the first integrals give two points on the (ℓ, h)-plane, $P_-(0, -2)$ and $P_+(0, 2)$. If in (3.6) we take $y = 0$, then the values of x_1, x_2 remain arbitrary up to the condition $x_1 x_2 = 4$. Therefore, we obtain the circle of degenerate neutral equilibria giving the point $P_0(0, 0)$ on the (ℓ, h)-plane. From the physical point of view this set of equilibria consists of all body's configurations such that the equatorial plane of the top coincides with the plane $O\boldsymbol{\alpha}\boldsymbol{\beta}$ and the latter basis is of the opposite orientation to the basis $O\boldsymbol{e}_1\boldsymbol{e}_2$; the rotating moment of the forces $\boldsymbol{e}_1 \times \boldsymbol{\alpha} + \boldsymbol{e}_2 \times \boldsymbol{\beta}$ is then identically zero (compare with the results of [17] for three constant fields).

To find the critical points of L, write equations (3.5) with $f = L$:

$$\begin{aligned} &x_2 z_1 - y_2 z_2 = 0, \qquad y_1 z_1 - x_1 z_2 = 0, \qquad x_1 x_2 - y_1 y_2 = -4, \\ &2n w_1 + (y_1 z_1 - x_1 z_2) w_3 = 0, \qquad 2n w_2 + (y_2 z_2 - x_2 z_1) w_3 = 0. \end{aligned}$$

This system together with (3.2) yields

$$w_1 = w_2 = 0, \qquad z_1 = z_2 = 0, \qquad x_1 = x_2 = 0, \qquad y_1 y_2 = 4.$$

Let $y_1 = 2\exp(-\mathrm{i}\psi)$ and $y_2 = 2\exp(\mathrm{i}\psi)$. Then from (3.1), (2.1) we have

$$\begin{aligned} &\boldsymbol{\alpha} = \boldsymbol{e}_1 \cos\psi - \boldsymbol{e}_2 \sin\psi, \qquad \boldsymbol{\beta} = \boldsymbol{e}_1 \sin\psi + \boldsymbol{e}_2 \cos\psi, \\ &\omega_1 = \omega_2 = 0, \qquad \omega_3 = \dot{\psi}, \qquad \ddot{\psi} = -2\sin\psi. \end{aligned} \tag{3.7}$$

This system describes the pendulum type motions about the axis $O\boldsymbol{\gamma} = O\boldsymbol{e}_3$. The corresponding values of the integrals are

$$\ell = 0, \qquad h = \tfrac{1}{2}\omega_3^2 - 2\cos\psi \geqslant -2.$$

Note that the set of points satisfying (3.7) includes two non-degenerate equilibria, but does not contain the circle of degenerate equilibria. Therefore, the set $\mathcal{C}^0$ consists of exactly two points of the phase space P^6. It is natural to expect the existence of non-trivial motions with $\boldsymbol{\gamma} \equiv -\boldsymbol{e}_3$ in addition to the set of degenerate equilibria. We see that this family of motions, if exists, is not critical for either of the first integrals H, L. We consider it in the next section.

4. Generic critical motions and integral values

The points of the set $\mathcal{C}$ not critical for either of the integrals H, L form critical motions of the generic type. These motions are periodic trajectories of system (2.1) obtained as the orbits of the action g_τ with $\tau = \sigma t$ ($\sigma = \text{const}$). Therefore, the corresponding expressions (3.7) give the analytical solutions for this type of motions. This part of $\mathcal{C}$ is described by system (3.5) with $f = H - \sigma L$. The first three equations give

$$w_1 = -\tfrac{1}{2}(y_1 z_1 - x_1 z_2)\sigma, \qquad w_2 = -\tfrac{1}{2}(y_2 z_2 - x_2 z_1)\sigma, \qquad w_3 = -\tfrac{1}{4}(x^2 - y^2 + 4)\sigma. \tag{4.1}$$

Eliminating w_j in the remaining equations, we obtain

$$y_1 = y_2, \tag{4.2}$$

$$x_1 z_2 u - (y_1 u - 8) z_1 = 0, \qquad x_2 z_1 u - (y_2 u - 8) z_2 = 0. \tag{4.3}$$

Here for brevity we put

$$u = [4 - (n-1)(x^2 - y^2)]\sigma^2. \tag{4.4}$$

Note that according to (4.2) and (3.3), $y = y_1 = y_2$.

First, suppose that $z_1 = z_2 = 0$. Then from (4.1) we get $w_1 = w_2 = 0$ and it follows from (3.4) that either $x = 0$ or $y = 0$. If $x = 0$, then $y^2 = 4$ and the last equation (4.1) gives $w_3 = 0$. These are the two non-degenerate equilibria studied above. If, in turn, $x \neq 0$, then $y = 0$, $x = 2$. The component w_3 remains arbitrary. For the variables in (2.1), we have $\boldsymbol{\gamma} = -\boldsymbol{e}_3$, $\boldsymbol{\omega} = 2\sigma \boldsymbol{e}_3$. These motions are permanent rotations about the third inertia axis, which stays orthogonal to the plane of the forces while $O\boldsymbol{e}_1\boldsymbol{e}_2$ and $O\boldsymbol{\alpha\beta}$ define opposite orientations. The values of the first integrals $h = 2\sigma^2$, $\ell = 4\sigma$ fill the parabola $h = \ell^2/8$.

Now consider the case $z_1 z_2 \neq 0$. Express the first integrals in variables (3.1):

$$L = \frac{n}{4}[(x_2 z_1 - y_2 z_2)w_1 + (x_1 z_2 - y_1 z_1)w_2] - \frac{1}{4}(x^2 - y^2 + 4)w_3,$$
$$H = \frac{1}{2}\left(n w_1 w_2 + w_3^2\right) - \frac{1}{2}(y_1 + y_2).$$

Then from (4.1), (4.2), (3.4) we get the values

$$\begin{aligned} \ell &= \frac{\sigma}{16}\{16 + 8[(n+1)x^2 + (n-1)y^2] - (2n-1)(x^2 - y^2)^2\}, \\ h &= -y + \frac{\sigma}{2}\ell, \\ y &= \pm(2 - x), \qquad x \in [0, 2]. \end{aligned} \tag{4.5}$$

Non-zero solutions of (4.3) with respect to z_1, z_2 exist if

$$[(x+y)u - 8][(x-y)u + 8] = 0.$$

Hence, eliminating u in (4.4), we obtain

$$\sigma^2 = \frac{\operatorname{sgn} y}{n - (n-1)x} \qquad \text{or} \qquad \sigma^2 = \frac{\operatorname{sgn} y}{(1-x)[n - (n-1)x]}.$$

These expressions together with (4.5) define ℓ, h on the bifurcation diagram as functions of one parameter x. The admissible values of x are cut from the basic segment $[0, 2]$ by the corresponding condition $\sigma^2(x) \geqslant 0$.

5. Bifurcation diagram

Denote

$$\begin{aligned} &\varphi_0(x) = x[2n - (n-1)x], \\ &\varphi_1(x) = n - (n-1)x, \qquad \varphi_2(x) = (1-x)\varphi_1(x), \\ &h_1(x) = -\frac{5}{2} + \frac{3}{2}x + \frac{n+x}{2\varphi_1(x)}, \qquad h_2(x) = -\frac{5}{2} + x + \frac{n+x}{2\varphi_2(x)}. \end{aligned} \tag{5.1}$$

The next theorem summarizes the above results.

Theorem. *The bifurcation diagram* Σ *of the momentum map for the dynamically symmetric top in* S^1*-symmetric pair of constant fields consists of the following subsets in the* (ℓ, h)*-plane:*

$$\delta_0 = \{P_-, P_+, P_0\}, \qquad \delta_1 = \{\ell = 0 : h \geqslant -2\},$$
$$\delta_2 = \left\{h = \frac{1}{8}\ell^2 : \ell \in \mathbb{R}\right\},$$
$$\delta_3 = \left\{\ell = \pm\frac{\varphi_0(x)}{\sqrt{\varphi_1(x)}}, h = h_1(x) : x \in I_3\right\},$$
$$\delta_4 = \left\{\ell = \pm\frac{\varphi_0(x)}{\sqrt{\varphi_2(x)}}, h = h_2(x) : x \in I_4\right\},$$
$$\delta_5 = \left\{\ell = \pm\frac{\varphi_0(x)}{\sqrt{-\varphi_1(x)}}, h = -h_1(x) : x \in I_5\right\},$$
$$\delta_6 = \left\{\ell = \pm\frac{\varphi_0(x)}{\sqrt{-\varphi_2(x)}}, h = -h_2(x) : x \in I_6\right\},$$

where

$$I_3 = \begin{cases} [0,2], & n < 2 \\ \left[0, \dfrac{n}{n-1}\right], & n \geqslant 2 \end{cases}, \qquad I_4 = \begin{cases} [0,2), & n \leqslant 2 \\ [0,1) \cup \left(\dfrac{n}{n-1}, 2\right], & n > 2 \end{cases},$$
$$I_5 = \begin{cases} \emptyset, & n \leqslant 2 \\ \left(\dfrac{n}{n-1}, 2\right], & n > 2 \end{cases}, \qquad I_6 = \begin{cases} (1,2], & n < 2 \\ \left(1, \dfrac{n}{n-1}\right), & n \geqslant 2 \end{cases}. \tag{5.2}$$

Obviously, $\delta_0 \subset \delta_1$. Despite this fact, we emphasize the set δ_0 of three points generated by the body equilibria. Note that the parameter x on the curves δ_3–δ_6 is equal to the value $\sqrt{(\alpha_1 - \beta_2)^2 + (\alpha_2 + \beta_1)^2}$ constant along each critical trajectory. Therefore, this expression can be taken as the partial integral generating the set of critical motions.

6. Examples of diagrams with respect to the physical parameter

The system considered, its momentum map and the bifurcation diagram described by the above theorem depend on one dimensionless parameter n expressing the ratio of two different principal moments of inertia. It follows immediately from expressions (5.2) for the admissible segments of x that the value $n = 2$ corresponding to the integrable case of Yehia separates principally different types of diagrams. In the case $n < 2$, the diagram does not change qualitatively. Even in the case $n = 1$, when some obvious degenerations take place in (5.1), topologically the diagram is the same as at close enough values of n. A typical diagram for $n < 2$ is shown in figure 1 (due to the symmetry with respect to the h-axis, we illustrate only the part $\ell \geqslant 0$).

For the values $n > 2$ there exist several types of bifurcation diagrams. They differ by the number of knots (points of self-intersection of the smooth segments, cusps, etc.) and chambers (connected components of the set $\mathbb{R}^2\backslash\Sigma$). Two diagrams for the case $n > 2$ along with the enlarged fragments are shown in figure 2. Here we take as typical examples the values (*a*) $n = 2.3$; (*b*) $n = 4$.

The main separating values of n can be established analytically. Consider, for example, the point Q_1 of the intersection of the curve δ_2 with the first branch of δ_4 ($x \in [0, 1)$). It is

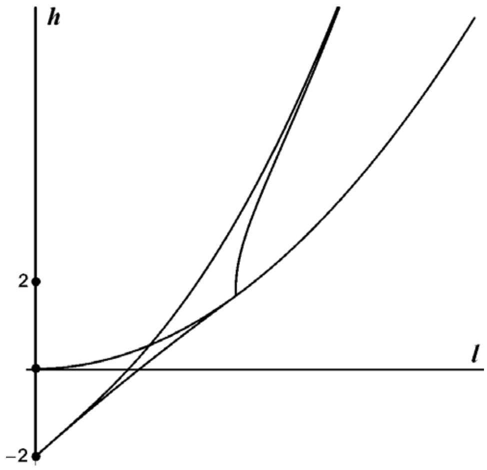

Figure 1. The bifurcation diagram for $n < 2$.

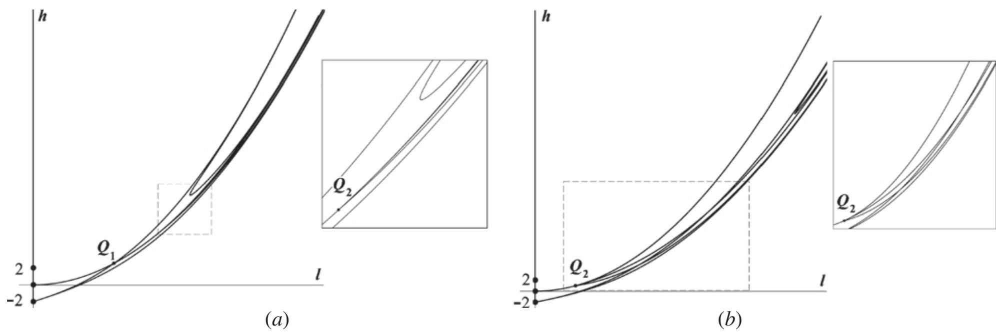

Figure 2. The bifurcation diagrams for $n > 2$.

shown in figure 2(*a*). One can see that during the passage from the chosen case (*a*) to the case (*b*), the point Q_1 disappears. Denote by n_* the corresponding separating value of the parameter n. To find n_*, note that on δ_4 the intersection with δ_2 is defined by the root of the polynomial

$$P(x) = (n-1)^3x^3 - 2(n-1)(n+5)x^2 + 4(5n-2)x - 8n$$

on the half-interval $[0, 1)$. Since $P(0) = -8n$ and $P(1) = -(n+1)(n-3)$, the intersection exists for all $n < n_* = 3$. The resultant of $P(x)$ and $P'(x)$ equals $256(n-1)^4(n-3)(n^4 - 5n^3 + 18n^2 + 2n + 11)$ and does not vanish if $n > 3$. Therefore, in this range of n, $P(x)$ has a unique real root which is always greater than 1. Thus, the intersection point cannot appear again for $n > 3$.

In figure 2 we also note that point Q_2 (the meeting point of three curves δ_2, δ_5 and the second branch of δ_4) crosses the first branch of δ_4 at some n. In fact, for such n we have $Q_2 = Q_1$. The coordinates of Q_2 are easily found from the equations of δ_4, δ_5 with $x = 2$:

$$l = \frac{4}{\sqrt{n-2}}, \qquad h = \frac{2}{n-2}.$$

Suppose $Q_2 \in \delta_4$ for $x \neq 2$; eliminate x to obtain the equation in n

$$n^4 - 3n^3 - 5n^2 + 20n - 11 = 0,$$

which has exactly four real roots. The separating case $Q_2 = Q_1$ is generated by the largest root $n \approx 2.538$.

Finally, in addition to the value $n = 2$, which corresponds to the globally integrable Yehia case, we find two more values of n generating the topological transformations of the bifurcation diagrams. In particular, the singular value is $n = 3$. It is possible that this fact reflects the existence of some partial integrability.

7. Summary

In this work we present the family of mechanical systems with three degrees of freedom admitting the S^1-symmetry with non-empty set of singular points. We obtain the equations of the bifurcation diagrams of the corresponding momentum maps. Some examples of the diagrams transformations are given, which show that the problem of complete classification of the diagrams with respect to the essential dimensionless parameter n can be non-trivial.

It follows from the works [11, 18, 19] that further generalizations exist for the motion of a dynamically symmetric gyrostat in the double homogeneous force field. The bifurcation diagrams will then depend on two physical parameters.

If we exclude from the phase space the critical stratified manifold $L = 0$, then the remaining system is reducible to the family of systems with two degrees of freedom marked by the constant non-zero value of L. The configuration space of each such system (the punctured two-dimensional sphere) is diffeomorphic to $\mathbb{R}^2$. The reduced potential is easily found using, for example, the Euler angles. To obtain the description of the integral manifolds, one needs to calculate the indices of all its critical points. The critical points themselves are in fact found above, but the topological analysis is not considered here. It is planned to present in the future for the gyrostat having this kind of symmetry.

In the case of Yehia ($n = 2$), the initial system admits one more integral K found by Bogoyavlensky [13] and generalizing the Kowalevski integral. Therefore, it is interesting to investigate the bifurcation diagram of the integral map

$$H \times L \times K : P^6 \to \mathbb{R}^3. \tag{7.1}$$

This diagram should be obtained as a degeneration of the general diagrams of the Kowalevski–Reyman–Semenov-Tian-Shansky top [18] built in [12, 14] according to the arising connection of the generalized area integral with the integrals H and L. Note that this degeneration is not straightforwardly obtained. Up to the present moment the only result here deals with the intersection of the Yehia case and the partial integrable case pointed out by Bogoyavlensky on the invariant manifold $M^4 = \{K = 0\}$. Then the extra partial Bogoyavlensky integral coincides with the Yehia integral. The bifurcation diagram of the restriction of the map $H \times L$ to M^4 and the phase topology of the induced system were studied in the work [20]. It was shown that in this case M^4 is not smooth. The non-smooth integral manifolds with self-intersections were revealed. The investigation of the map (7.1) will give the possibility of describing the three-dimensional Liouville foliation in the Yehia case, which includes the non-trivial integral manifold transformations found in [20] as bifurcations inside critical subsystems.

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